initial, left- and right-side model boundary voltages, respectively; r, c_e, ohmic resistance and capacitance of the model cell; R_g, R_c, left- and right-side model boundary resistances; n, number of model cells; k_l, k_τ, k_T, coordinate, time, and temperature scales; x_e, cell coordinate; τ_e, time.

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USE OF A HYPERBOLIC EQUATION IN THERMAL-CONDUCTIVITY THEORY

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A solution of the telegraph equation is given which is close to a self-similar solution.

1. Singularities in Solutions of the Classical Equation of Thermal Conductivity. In the simulation of thermal processes by the equation of thermal conductivity,

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}. \tag{1}$$

certain singularities occur. Actually, we consider the fundamental solution of Eq. (1)

$$T_0(x, t) = 1/\sqrt{4\pi at}\exp[-x^2/(4at)] \tag{2}$$

and find the mean value of the square of the temperature displacement from its initial position during the time t:

$$\overline{\Delta x^2} = \int (x - x_0)^2 T_0(x, t)\,dx / \int T_0(x, t)\,dx = 2at. \tag{3}$$

We define the rate of temperature displacement in the following manner:

$$V = \frac{d}{dt}\left(\sqrt{\overline{\Delta x^2}}\right) = \sqrt{a/(2t)}. \tag{4}$$

It then follows that the temperature nonuniformity is propagated instantaneously at the initial time. A similar paradox occurs in the theory of Brownian motion [1].

Using the fundamental solution, we find an equation for the surface of maximum temperature. To do this, we differentiate Eq. (2) with respect to time and equate the result to zero. Then $x^2 - 2at = 0$, hence $x = \sqrt{2at}$ and

$$V_{max} = \frac{dx}{dt} = \sqrt{a/(2t)}, \tag{5}$$

i.e., the expression for the rate of displacement of the surface of maximum temperature agrees with Eq. (4) and V_{max} has a marked singularity.

The use of the classical equation of thermal conductivity in phase-transition problems also leads to a similar paradox. Actually, in the well-known Stefan solution [2], the law of motion for the freezing line has the form $z = p\sqrt{t}$ so that

$$V_z = \frac{dz}{dt} = p/(2\sqrt{t}).$$

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Note that some problems in the theory of filtration and in vortex motion reduce to boundary-value problems for Eq. (1); consequently, the singularities mentioned also occur in these problems.

The existence of these paradoxes points to the need for refinement of the classical equation of thermal conductivity.

On the basis of isotherm analysis, the following equation was obtained [9]:

$$c\Big/\left(-\frac{\partial c}{\partial t}\right)\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}=c^3\Big/\left(-\frac{\partial c}{\partial t}\right)\frac{\partial^2 T}{\partial x^2}. \tag{6}$$

We introduce the following notation: $\lambda = c/(-dc/dt)$, $a = c^2\lambda$. In some cases, λ and a can be considered considered constants. Equation (6) then takes the form

$$\lambda\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}=a\frac{\partial^2 T}{\partial x^2}. \tag{7}$$

A similar equation was obtained by many authors [3, 7, 8] on the basis of relaxation concepts. As shown in [9], the problem for wave propagation of heat is incorrectly formulated if a parabolic equation is used in it. The incorrectness is eliminated by conversion to Eq. (7).

2. A Solution of the Telegraph Equation. We find a solution of Eq. (7) which transforms into the fundamental solution of the parabolic equation of thermal conductivity when $\lambda \to 0$. To do this, we introduce the new variables

$$\tau = t,\ \eta = x^2/\tau. \tag{8}$$

We seek a solution of Eq. (7) in the form

$$T=\tau^{-\frac{1}{2}}f(\eta,\tau). \tag{9}$$

We indicate the conversion formulas

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}-\tau^{-1}\eta\frac{\partial}{\partial \eta};\ \frac{\partial}{\partial x}=2\tau^{-\frac{1}{2}}\eta^{\frac{1}{2}}\frac{\partial}{\partial \eta}.$$

We now rewrite Eq. (7) in the form

$$\lambda\left[\frac{3}{4}\tau^{-1}f-\frac{\partial f}{\partial \tau}+\tau\frac{\partial^2 f}{\partial \tau^2}+3\tau^{-1}\eta\frac{\partial f}{\partial \eta}+\tau^{-1}\eta^2\frac{\partial^2 f}{\partial \eta^2}\right.$$
$$\left.-2\eta\frac{\partial^2 f}{\partial \tau\partial \eta}\right]-\frac{1}{2}f+\tau\frac{\partial f}{\partial \tau}-\eta\frac{\partial f}{\partial \eta}=a\left[2\frac{\partial f}{\partial \eta}+4\eta\frac{\partial^2 f}{\partial \eta^2}\right]. \tag{10}$$

We seek a solution of Eq. (10) in the form

$$f(\eta,\tau)=\sum_{n=0}^{\infty}\tau^{-n}f_n(\eta). \tag{11}$$

Substituting Eq. (11) in Eq. (10) and equating coefficients of identical powers of τ on the right and left sides of the resultant equality, we arrive at a system of equations for f_n:

$$-\frac{1}{2}f_0-\eta f_0'=a(2f_0'+4\eta f_0''),$$

$$\lambda\left[\frac{3}{4}f_0+3\eta f_0'+\eta^2 f_0''\right]-\frac{3}{2}f_1-\eta f_1'=a(2f_1'+4\eta f_1''),$$

. . . .

$$\lambda\left[\left(n^2-\frac{1}{4}\right)f_{n-1}+\eta(2n+1)f_{n-1}'+\eta^2 f_{n-1}''\right]+\left(n+\frac{1}{2}\right)f_n-\eta f_n'=a(2f_n'-4\eta f_n'').$$

The solution of the equation for f_0 has the form

$$f_0(\eta)=A\exp[-\eta/(4a)].$$

The functions $f_n(\eta)$ $(n \geq 1)$ should be sought in the form

$$f_n(\eta) = A\lambda^n \exp[-\eta/(4a)] \sum_{i=0}^{2n} a_{ni}\eta^i,$$

where the a_{ni} are unknown coefficients.

We obtain the general form of the desired solution $T(\eta, \tau)$ by substituting in Eq. (9) the expansion in terms of inverse powers of τ found for $f(\eta, \tau)$. Returning to the variables x and t, we write T(x, t) in the form

$$T(x, t) = A \exp[-x^2/(4at)]\, t^{-\frac{1}{2}} \sum_{n=0}^{\infty} (\lambda/t)^n \sum_{i=0}^{2n} a_{ni} \left(\frac{x^2}{t}\right)^i. \qquad (12)$$

We write down the values of some of the coefficients:

$$a_{00} = 1;\ a_{10} = \frac{3}{4} - 2ab_1;\ a_{11} = b_1;\ a_{12} = -\frac{1}{16a^2};\ a_{20} = \frac{75}{32} - 15ab_1 + 12a^2b_2;\ a_{21} = \frac{45}{4}b_1 - \frac{15}{16a} - 12ab_2;\ a_{22} = b_2;\ a_{23} = -\frac{1}{36a^3} - \frac{b_1}{16a^2};$$

$$a_{24} = \frac{1}{512a^4};\ a_{30} = \frac{1225}{128} - \frac{3465}{16}ab_1 + 315a^2b_2 - 120a^3b_3;\ a_{31} = -\frac{1050}{256a} + \frac{8295}{32}b_1 - 420ab_2 + 180a^2b_3;\ a_{32} = \frac{1155}{512a^2} - \frac{420}{16a}b_1 + \frac{105}{2}b_2 - 30ab_3;\ a_{33} = b_3;\ a_{34} = -\frac{43}{2048a^4} - \frac{7b_1}{256a^3} - \frac{b_2}{12a^2};\ a_{35} = \frac{1}{512a^5} + \frac{b_1}{512a^4};\ a_{36} = -\frac{1}{24576a^6}.$$

The coefficients A and b_j $(j = 1, 2, \ldots)$ are undetermined constants. In the following, we set $A = 1/\sqrt{4\pi a}$. We consider the functions

$$\psi(x, t) = \begin{cases} 0 & \text{for } \left|\frac{x}{\sqrt{a}}\right| > \frac{t}{\sqrt{\lambda}}, \\ I_0\left(\sqrt{\frac{t^2}{4\lambda^2} - \frac{x^2}{4\lambda a}}\right) & \text{for } \left|\frac{x}{\sqrt{a}}\right| \leqslant \frac{t}{\sqrt{\lambda}}, \end{cases}$$

$$\varphi(x, t) = \sqrt{\lambda}/(2a) \exp[-t/(2\lambda)] \frac{\partial}{\partial t}[\psi(x, t)]. \qquad (13)$$

$I_0(z)$ is a modified Bessel function [3]. The derivatives at the points $|x/\sqrt{a}| = t/\sqrt{\lambda}$ should be understood in the generalized sense [4], namely:

$$\varphi(x, t)\Big|_{\frac{x}{\sqrt{a}} = \frac{t}{\sqrt{\lambda}}} = \frac{1}{2a}\exp[-t/(2\lambda)]\,\delta\left(\frac{x}{\sqrt{a}} - \frac{t}{\sqrt{\lambda}}\right),$$

$$\varphi(x, t)\Big|_{\frac{x}{\sqrt{a}} = -\frac{t}{\sqrt{\lambda}}} = \frac{1}{2a}\exp[-t/(2\lambda)]\,\delta\left(\frac{x}{\sqrt{a}} + \frac{t}{\sqrt{\lambda}}\right).$$

It is easy to obtain an asymptotic expansion of $\varphi(x, t)$ for $|x/\sqrt{a}| < t/\sqrt{\lambda}$ in the form (12). In this case, $b_1 = -1/2$, $b_2 = -3/4$, ... Note that the function $\varphi(x, t)$ satisfies Eq. (7). Calculating the rate of propagation of the initial maximum temperature, we find that $V_{max} = \sqrt{a/\lambda}$; i.e., the paradox noted is eliminated in the conversion from Eq. (1) to Eq. (7).

With the help of the function $\varphi(x, t)$ we now construct a solution of Eq. (7) satisfying the condition T(x, 0) = F(x). Proceeding as in [5], we consider the superposition of solutions

$$T(x, t) = \sum_i F(x_i)\,\varphi(\Delta x_i, t)\,\Delta x_i. \quad (14)$$

Converting to an integral in Eq. (14), we obtain

$$T(x, t) = \int_{-\infty}^{\infty} F(z)\,\varphi(x - z, t)\,dz. \quad (15)$$

Analysis of this solution shows that Eq. (15) transforms into the Poisson integral for the classical equation of thermal conductivity [6] both for $t \to \infty$ and for $\lambda \to 0$. Considering the specific form of the function $\varphi(x, t)$, Eq. (15) can also be written in the form

$$T(x, t) = \exp[-t/(2\lambda)]/2a\left\{F\left(\frac{x}{\sqrt{a}} - \frac{t}{\sqrt{\lambda}}\right) + F\left(\frac{x}{\sqrt{a}} + \frac{t}{\sqrt{\lambda}}\right)\right.$$

$$\left. + \sqrt{\lambda}\int_{x/\sqrt{a}-t/\sqrt{\lambda}}^{x/\sqrt{a}+t/\sqrt{\lambda}} F(z)\frac{\partial}{\partial t}\left[I_0\left(\sqrt{t^2/(4\lambda^2) - (x-z)^2/(4\lambda a)}\right]dz.\right.$$

In the case where F(z) is of complicated form and it is difficult to compute the integral on the right side of Eq. (15), it is advisable to use the asymptotic expansion of the function $\varphi(x, t)$.

NOTATION

T, temperature; t, time; x, spatial coordinate; a, thermal diffusivity; c, rate of isotherm displacement.

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